Modeling and simulation of floating structures

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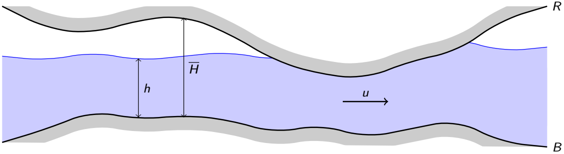
Congested shallow water model

Starting from **Navier-Stokes** with gravity and **roof**, we get [Gerbeau, Perthame'00]

$$\begin{cases} \partial_t h + \nabla \cdot (hu) = 0 \\ \partial_t (hu) + \nabla \cdot (hu \otimes u) = -h\nabla \left(g\left(h+B\right)+p\right) \\ h \leq \overline{H} \qquad \left(h-\overline{H}\right)\left(p-P_a\right) = 0 \qquad p \geq P_a \end{cases}$$



Energy conservation For smooth enough solutions, the following energy law holds

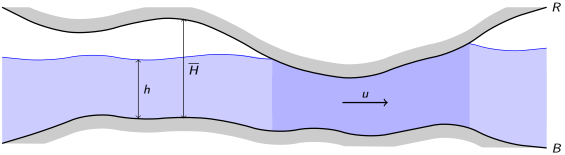
$$\partial_t \mathcal{E} + \nabla \cdot \mathcal{G} = -\left(p-P_a\right)\partial_t \overline{H} + h\partial_t\left(gB+P_a\right)$$

with the mechanical energy $\mathcal{E} = \frac{h}{2}\|u\|^2 + gh\left(B+\frac{h}{2}\right) + hP_a$ and the energy flux $\mathcal{G}$

We have a "**hyperbolic** system" which has to verify a **constraint**

$$\begin{cases} \partial_t h + \nabla \cdot (hu) = 0 \\ \partial_t (hu) + \nabla \cdot (hu \otimes u) = -h\nabla \left(g\left(h + B\right) + p\right) \\ h \leq \overline{H} \qquad \left(h - \overline{H}\right)\left(p - P_a\right) = 0 \qquad p \geq P_a \end{cases}$$

Difficulty: $\overline{H}$ depends on **space** and **time** and **interaction** between water and roof



> COUPLING STRATEGY [Lannes'17]
  - Dynamic of the interface?
  - Transmission conditions at the interface?

> UNIFIED STRATEGY [Bourdarias *et al.*'12]

Pseudo-compressible relaxation approach

Relaxed model

$$\begin{cases} \partial_t h_\lambda + \nabla \cdot (h_\lambda u_\lambda) = 0 \\ \partial_t (h_\lambda u_\lambda) + \nabla \cdot (h_\lambda u_\lambda \otimes u_\lambda) = -h_\lambda \nabla \phi_\lambda \\ p_\lambda = g \dfrac{\left(h_\lambda - \overline{H}\right)_+}{\lambda^2} + P_a \end{cases}$$

with $\phi_\lambda = g(h_\lambda + B) + p_\lambda$

Difficulty: relevant for $\lambda \ll 1$

Energy conservation For smooth enough solutions, the following energy law holds

$$\partial_t \mathcal{E}_\lambda + \nabla \cdot \mathcal{G}_\lambda = -(p_\lambda - P_a)\,\partial_t \overline{H} + h_\lambda \partial_t (gB + P_a)$$

with $\mathcal{E}_\lambda = \dfrac{h_\lambda}{2}\|u_\lambda\|^2 + g h_\lambda \left(B + \dfrac{h_\lambda}{2}\right) + h_\lambda P_a + g\dfrac{\left(h_\lambda - \overline{H}\right)_+^2}{\lambda^2}$ and the energy flux $\mathcal{G}_\lambda$

Relaxed model

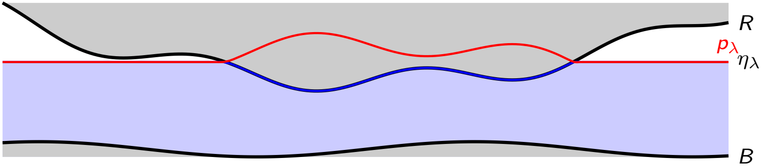
$$\begin{cases} \partial_t h_\lambda + \nabla \cdot (h_\lambda u_\lambda) = 0 \\ \partial_t (h_\lambda u_\lambda) + \nabla \cdot (h_\lambda u_\lambda \otimes u_\lambda) = -h_\lambda \nabla \phi_\lambda \\ p_\lambda = g \dfrac{\left(h_\lambda - \overline{H}\right)_+}{\lambda^2} + P_a \end{cases}$$

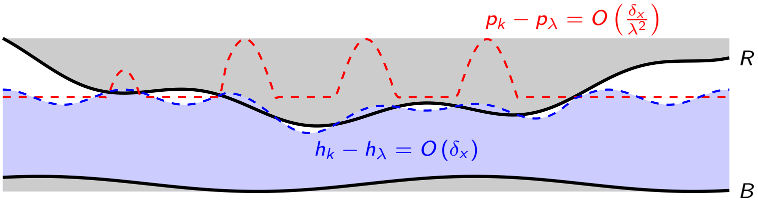
Hyperbolicity The model is strictly hyperbolic with the eigenvalues

$$0 \text{ and } u_\lambda \pm \sqrt{\left(1 + \frac{\mathbb{1}_{h_\lambda \geq \overline{H}}}{\lambda^2}\right) g h_\lambda}$$

$\Rightarrow$ Use a scheme accurate in the **limit** where **potential forces are large in front of the advection termes**

Lake at rest preservation: steady solution with vanishing velocity





A perturbation on h induces a perturbation on p, induces a perturbation on $u, \ldots$

$\Rightarrow$ Use a **well-balanced** scheme

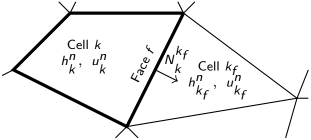
Numerical scheme

some references: [Dellacherie *et al.*'16], [Herbin *et al.*'14], [Parisot, Vila'16], ...

in cell: $\psi_k = \dfrac{1}{|V_k|}\displaystyle\int_{V_k} \psi \mathrm{d}x$

at edge: $2\,(\psi)_f = \psi_k + \psi_{k_f}$ and $2\,[\psi]_k^{k_f} = \psi_{k_f} - \psi_k$

parameters: $\ell_k = \dfrac{|V_k|}{|\partial V_k|}$ and $\mu_f^k = \dfrac{|f|}{|\partial V_k|}$



Step 1: **implicit** scheme of type **non-linear** advection-diffusion for the water height

$$h_k^{n+1} - h_k^n + \frac{\mathrm{d}t}{\ell_k} \sum_{f\in\mathbb{F}_k} \left(\left(h^{n+1} u^n\right)_f \cdot N_k^{k_f} - \mathrm{d}t \left(\frac{h^{n+1}}{\ell}\right)_f \left[\phi^{n+1}\right]_k^{k_f} \right) \mu_f^k = 0$$

Step 2: **explicit upwind** scheme with source term for the velocity

$$h_k^{n+1} u_k^{n+1} - h_k^n u_k^n \quad + \quad \frac{\mathrm{d}t}{\ell_k} \sum_{f\in\mathbb{F}_k} \left(u_k^n \left(\mathcal{F}_f^{n+1} \cdot N_k^{k_f}\right)_+ - u_{k_f}^n \left(\mathcal{F}_f^{n+1} \cdot N_k^{k_f}\right)_- \right) \mu_f^k$$
$$= -\mathrm{d}t \frac{h_k^{n+1}}{\ell_k} \sum_{f\in\mathbb{F}_k} \left[\phi^{n+1}\right]_k^{k_f} N_k^{k_f} \mu_f^k$$

Discrete energy

Under the **non-restrictive** CFL condition

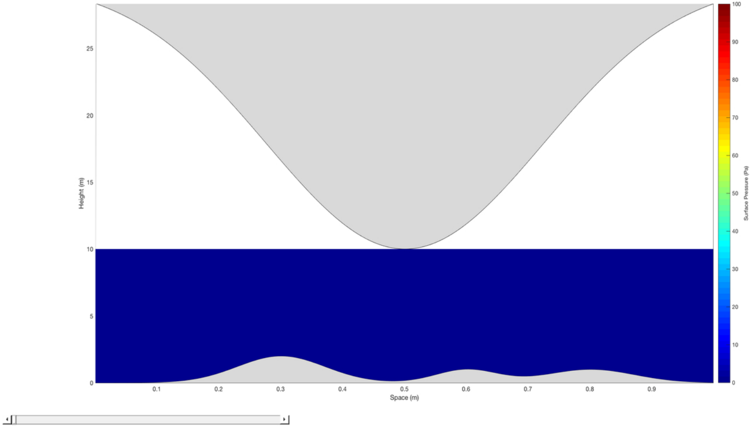
$$\left(\left|u_f^n \cdot N_f^k\right| + \sqrt{\frac{1}{2}}\sqrt{\left|[\phi^{n+1}]_k^{k_f}\right|}\right)\delta_t^{n+1} \leq \frac{\min\left(h_k^{n+1}, h_{k_f}^{n+1}\right)}{h_k^{n+1} + h_{k_f}^{n+1}} \min\left(l_k, l_{k_f}\right)$$

the scheme admits the following **energy dissipation law**

$$\partial_t^{n+1}\mathcal{E}_k + \frac{1}{l_k}\sum_{f \in \mathbb{F}_k} \mathcal{G}_f^{n+1}\mu_f^k \leq -\left(p_k^{n+1} - P_k^{n+1}\right)\partial_t^{n+1}\overline{H}_k + h_k^n \partial_t^{n+1}\left(gB_k + P_k\right)$$

with the discrete mechanical energy $\mathcal{E}_k^n = \mathcal{E}\left(h_k^n, u_k^n\right)$, the discrete flux of energy $\mathcal{G}_k^n$ and the discrete time derivative $\partial_t^{n+1}\psi = \dfrac{\psi^{n+1} - \psi^n}{\delta_t^{n+1}}$

Numerical results with fixed buoy



Buoy dynamics

Adding **Newton's second law of motions**

$$\begin{cases} \partial_t h + \nabla \cdot (hu) = 0 \\ \partial_t (hu) + \nabla \cdot (hu \otimes u) = -h \nabla (g (h + B) + p) \\ h \leq \overline{H} \qquad \left(h - \overline{H}\right)(p - P_a) = 0 \qquad p \geq P_a \\ M\ddot{\zeta} = -Mg + \int_{\Omega_x} (p - P_a)\, \mathrm{d}x \end{cases}$$

with $R(x,t) = R_0(x) + \zeta(t)$

Energy conservation For any smooth solution the following energy balance law holds

$$\partial_t \left(\int_{\Omega_x} \mathcal{E}\, \mathrm{d}x + E \right) = \int_{\Omega_x} (p - P_a)\, \partial_t B \mathrm{d}x + \int_{\Omega_x} h \partial_t (gB + P_a)\, \mathrm{d}x$$

where $E = \frac{M}{2}\dot{\zeta}^2 + Mg\zeta$ and $\mathcal{E} = \frac{1}{2} h \|u\|^2 + gh \left(B + \frac{h}{2}\right) + hP_a$

Discretization of the buoy dynamics equation

$$\ddot{\zeta}^{n+1} = -g + \frac{1}{M}\sum_{k\in\mathbb{T}} |k| \left(p_k^{n+1} - P_k^{n+1}\right)$$

Using a Newmark scheme, **the discrete energy law** writes

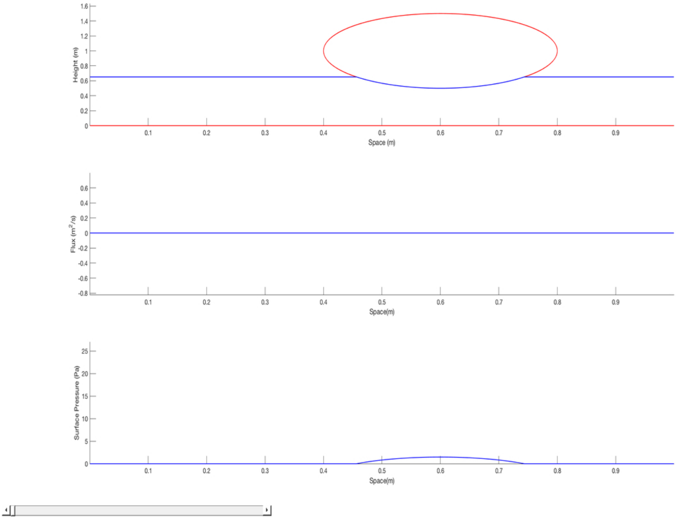
$$\begin{aligned} E^{n+1} - E^n \quad &= -\left(\alpha - \frac{1}{2}\right) M \frac{\beta-\alpha}{2} \left(\delta_t^{n+1}\right)^2 \left(\ddot{\zeta}^{n+1} - \ddot{\zeta}^n\right)^2 \\ &+ \left(\zeta^{n+1} - \zeta^n\right) \left(\sum_{k\in\mathbb{T}} \left(|k| \left(\alpha \left(p_k^{n+1} - P_k^{n+1}\right) - (\alpha-1)\left(p_k^n - P_k^n\right)\right)\right)\right) \end{aligned}$$

with $E^n = \frac{1}{2} M \dot{\zeta}^{n^2} + \frac{\beta-\alpha}{2} \frac{\left(\delta_t^{n+1}\right)^2}{2} M \ddot{\zeta}^{n^2} + Mg\zeta^n$.

Energy law for the coupled system Let $\alpha = \beta = 1$. Then the scheme admits the following dissipation law

$$\partial_t^{n+1}\left(\sum_{k\in\mathbb{T}} |k|\mathcal{E}_k + E\right) \leq \sum_{k\in\mathbb{T}} \left(|k| h_k^n \partial_t^{n+1}\left(gB_k + P_k\right)\right) + \sum_{k\in\mathbb{T}} \left(|k| \left(p_k^{n+1} - P_k^{n+1}\right) \partial_t^{n+1} B_k\right)$$

Numerical result with buoy dynamics



Conclusion

- Derivation of a shallow water type model for **partially free surface** flow
- **Relaxed model** introduced
- **Numerically approaching the non-constant constraint**

Perspectives

- Analysis: **convergence** when $\lambda \to 0$, **non-conservative product** $h\nabla p$
- Validation: confrontation with **real life data** HYDROTUBE ENERGIE
- Modeling: more **dynamics** for buoy, more **physical** flow, **air** modeling, **submerged** object

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